The lower boundary of the accretion column in magnetic cataclysmic variables

Kinwah Wu[1,2] and Mark Cropper[2]

[1] *Research Centre for Theoretical Astrophysics, School of Physics A28, University of Sydney, NSW 2006, Australia*
[2] *Mullard Space Science Laboratory, University College London, Holmbury St Mary, Dorking, Surrey RH5 6NT*

ABSTRACT
Using a parameterised function for the mass loss at the base of the post-shock region, we have constructed a formulation for magnetically confined accretion flows which avoids singularities, such as the infinity in density, at the base associated with all previous formulations. With the further inclusion of a term allowing for the heat input into the base from the accreting white dwarf we are able also to obtain the hydrodynamic variables to match the conditions in the stellar atmosphere. (We do not, however, carry out a mutually consistent analysis for the match). Changes to the emitted X-ray spectra are negligible unless the thickness of mass leakage region at the base approaches or exceeds one percent of the height of the post-shock region. In this case the predicted spectra from higher-mass white dwarfs will be harder, and fits to X-ray data will predict lower white-dwarf masses than previous formulations.

Key words: accretion, accretion discs – cataclysmic variables – hydrodynamics – shock waves – stars: binaries: close – stars: white dwarfs – X-rays: stars

1 INTRODUCTION

In magnetic cataclysmic variables (mCVs) the accreting material from the secondary star is entrained onto magnetic field lines and accretes onto the surface of the white dwarf. There it forms a standoff shock, followed by a hot post-shock region of plasma settling onto the white dwarf surface as it cools, principally by bremsstrahlung X-ray radiation, and by optical/IR cyclotron radiation if the magnetic field is sufficiently strong (see Cropper 1990 and Warner 1995 for reviews of mCVs).

There have been a number of studies of the post-shock accretion flow (e.g. Aizu 1973; Langer, Chanmugam & Shaviv 1981; Chevalier & Imamura 1982; Kylafis & Lamb 1982; Chanmugam, Langer & Shaviv 1985; Imamura et al. 1987; Wu 1994; Wu, Chanmugam & Shaviv 1994; Woelk & Beuermann 1996; Kocabiyik 1997; Cropper et al. 1999; Saxton & Wu 1999), investigating different aspects of the region, both analytically and numerically (see Wu 2000 for a review). Despite the fact that substantial progress has been made, the existing studies generally assume that the velocity and the temperature drop to zero at the base of the accretion column. The assumption of zero flow velocity and the strict requirement of mass continuity along the field lines immediately implies that the matter density at the base of the accretion column must reach infinity. Recent work by Cropper, Wu & Ramsay (2000) has highlighted the consequences of an infinite density at the base of the accretion column: the X-rays from the shock-heated region is emitted mostly from its base where the density is rising steeply. This is true at energies up to approximately that of the shock temperature, which is in the $\sim 10 - 60$ keV range — higher than most imaging or CCD-based X-ray instruments. The preponderance of emission from the base of the accretion column has made a more in-depth understanding of the physical conditions in shock-heated emission regions essential, not only from the theoretical point of view but also for modelling and extracting information from the observed X-ray spectra.

Several aspects have to be considered in order to derive a self-consistent formulation of the accretion flow, especially at the boundary layer where the hydrodynamic flow merges into a hydrostatic white-dwarf atmosphere. In the stationary-state case, the hydrodynamic variables at the base of the accretion column should match smoothly to the corresponding variables at the white-dwarf atmosphere. In addition, the energy deposition due to the heat flux emerging from below the white-dwarf atmosphere should be considered. Moreover, at some point in the post-shock region, material may become sufficiently cold that a fraction becomes neutral and cannot be efficiently confined by the magnetic field. In the time-dependent situation, it is necessary to consider the response of the white-dwarf atmosphere to changes in the local mass-accretion rate and its effects on the stability of the accretion flow.

In this paper we investigate the stationary-state accretion onto magnetic white dwarfs, with emphasis on the post-shock flow at the boundary layer above the white-dwarf atmosphere. We consider a modification to the conventional hydrodynamic formulation so that matter diffusion across the field lines is allowed in a thin region at the bottom of the accretion column.

In addition, the heat flux from the white-dwarf atmosphere into the accretion column is considered. As it is an exploratory study, we neglect the complications such as geometrical, gravity and two-fluid effects. We demonstrate that a closed-form solution as that described in Aizu (1973) and Wu et al. (1994) (see also Chevalier & Imamura 1982) can be obtained for this formulation. Moreover, the infinities and discontinuities at the base, which lead to unphysical observational consequences, are eliminated. We calculate the emission from the modified post-shock region and compare their spectral properties with those of the conventional post-shock region.

2   STATIONARY-STATE ACCRETION FLOW

2.1   Modified hydrodynamic formulation

We assume that the flow of the ionised accreting material follows the magnetic field lines and that the field lines are perpendicular to the white-dwarf surface, so the flow is (quasi-)one-dimensional. We neglect gravity effects, which are important when the shock height $x_{\rm s}$ is not negligible in comparison with the white-dwarf radius $R_{\rm w}$ (see Cropper et al. 1999). We also assume that only the ionised matter is strictly confined by the magnetic field, while the cold neutral atomic matter can "leak" out of the accretion column. The accretion matter obeys the ideal gas law, i.e. $P = 2\rho kT/m_{\rm p}$ and $\gamma = 5/3$, where P is the gas pressure, T the gas temperature, k the Boltzmann constant, $m_{\rm p}$ the proton mass and γ the adiabatic index of the gas. At the lower boundary of the accretion column the temperature gradient and the bulk-flow velocity are zero, and the cooling rate equals the heating rate.

In our formulation the stationary state mass continuity, momentum and energy equations are

$$\frac{\partial}{\partial x}\rho v \;=\; -\Sigma \;, \tag{1}$$

$$\frac{\partial}{\partial x}P \,+\, \rho v\, \frac{\partial}{\partial x}v \;=\; v\Sigma \;, \tag{2}$$

$$v\, \frac{\partial}{\partial x}P \,+\, \gamma P\, \frac{\partial}{\partial x}v \;=\; -(\gamma-1)\left[\,\Lambda_{\rm c} - \Lambda_{\rm h} + \frac{1}{2}\,v^2\Sigma\,\right] \;, \tag{3}$$

where v is the flow velocity, and ρ the density. Σ is the sink term specifying the rate of mass loss out from the accretion column; $\Lambda_{\rm c}$ is the effective cooling function, and $\Lambda_{\rm h}$ is the heating function. In the construction of the formulation, we have implicitly assumed that the neutral matter, which is not confined by the magnetic field, transfers its momentum and energy to the ionised matter before it leaves the accretion column. Under this assumption momentum is conserved along each flow line of the ionised material and energy is dissipated only by emitting radiation. In reality the neutral matter may carry away substantial amount of momentum and energy, and hence the strict condition for conservation of momentum and energy along the field lines needs to be relaxed. This case will be discussed elsewhere (Wu & Cropper, in preparation).

The accretion shock is assumed to be strong and adiabatic, with the immediate post-shock velocity equal to a quarter of the pre-shock velocity, which is taken to be the free-fall velocity at the white-dwarf surface, i.e. $v = -v_{\rm ff}/4$ at $x = x_{\rm s}$. (Here and hereafter, the subscript "s" denotes variables at the shock.) The post-shock pressure is three-quarter of the pre-shock ram pressure of the inflowing gas, i.e. $P_{\rm s} = 3\rho_{\rm a}v_{\rm ff}^2/4$, where $\rho_{\rm a}$, the density of the pre-shock accretion matter, is related to the specific accretion rate (per unit area) $\dot{m}$ by $\rho_{\rm a} = \dot{m}/v_{\rm ff}$.

We define the dimensionless variables $\xi = x/x_{\rm s}$, $\zeta = \rho/\rho_{\rm a}$, $\tau = -v/v_{\rm ff}$ and $\varpi = P/\rho_{\rm a}v_{\rm ff}^2$, and substitute them into the mass, momentum and energy equations. In terms of these variables, the hydrodynamic equations are

$$\frac{\partial}{\partial \xi}\zeta\tau \;=\; \tilde{\Sigma} \;; \tag{4}$$

$$\frac{\partial}{\partial \xi}\varpi \,+\, \zeta\tau\, \frac{\partial}{\partial \xi}\tau \;=\; -\tau\tilde{\Sigma} \;; \tag{5}$$

$$\tau\frac{\partial}{\partial \xi}\varpi \,+\, \gamma\varpi\, \frac{\partial}{\partial \xi}\tau \;=\; (\gamma-1)\left[\,\tilde{\Lambda} + \frac{1}{2}\,\tau^2\tilde{\Sigma}\,\right] \;, \tag{6}$$

where $\tilde{\Sigma} = (x_{\rm s}/\rho_{\rm a}v_{\rm ff})\Sigma$, and $\tilde{\Lambda} = (x_{\rm s}/\rho_{\rm a}v_{\rm ff}^3)(\Lambda_{\rm c} - \Lambda_{\rm h})$. At the white-dwarf surface ($\xi = 0$), the velocity $\tau = 0$; at the shock ($\xi = 1$), $\tau = 1/4$ and the pressure $\varpi = 3/4$.

The sink term above, $\tilde{\Sigma}$, is not a direct observable, but its integration over the distance along the flow is the mass loss rate per unit area. It is therefore more appropriate to consider the integration of the sink term over the distance and express it in terms of the flow velocity, the independent variable that we use in solving the hydrodynamic equations.

In terms of the integration of the sink term over the distance we can define a dimensionless variable

$$\sigma(\tau) \;=\; 1 - \int_{\xi(\tau)}^{1} d\xi'\; \tilde{\Sigma}(\xi') \;. \tag{7}$$

At the white-dwarf surface $\sigma(0) = 0$, and at the shock $\sigma(1/4) = 1$. The variable $\sigma(\tau)$ clearly satisfies the mass continuity equation and it is in fact the dimensionless specific mass accretion rate in the presence of matter leakage.

The dimensionless density can now be obtained by solving the mass continuity equation, and it is

$$\zeta(\tau) \; = \; \frac{\sigma(\tau)}{\tau} \; . \tag{8}$$

The dimensionless pressure, obtained by integrating the momentum equation, is

$$\varpi(\tau) \; = \; 1 \; - \; \tau \, \sigma(\tau) \; . \tag{9}$$

Combining equations (8) and (9) with the energy equation yields

$$\frac{\partial}{\partial \xi} \tau \; = \; (\gamma - 1) \, \tilde{\Lambda} \left[\gamma - (\gamma + 1) \, \tau \sigma - \frac{1}{2} \, (\gamma + 1) \, \tau^2 \frac{\partial}{\partial \tau} \sigma \right]^{-1} \; . \tag{10}$$

The above equation can be integrated to obtain a closed-form expression for $\xi(\tau)$:

$$\xi(\tau) \; = \; 1 \; - \; \left(\frac{1}{\gamma - 1} \right) \int_{\tau}^{1/4} d\tau' \, \frac{1}{\tilde{\Lambda}} \left[\gamma \; - (\gamma + 1) \, \tau' \sigma - \frac{1}{2} \, (\gamma + 1) \, \tau'^2 \frac{\partial}{\partial \tau'} \sigma \right] \; . \tag{11}$$

We follow Wu (1994) and consider the cooling term $\Lambda_{\rm c}$ (in equation [3]) in a composite-functional form

$$\Lambda_{\rm c} \; = \; A \rho^2 \left(\frac{P}{\rho} \right)^{1/2} \left[1 + \epsilon_{\rm s} \left(\frac{P}{P_{\rm s}} \right)^{\alpha} \left(\frac{\rho_{\rm s}}{\rho} \right)^{\beta} \right] \; . \tag{12}$$

The parameter $\epsilon_{\rm s}$ specifies the ratio of the bremsstrahlung cooling time-scale to the cyclotron cooling time-scale at the shock. The value of the constant A is 3.9×10^{16} in c.g.s. units for pure hydrogen plasmas (see Rybicki & Lightman 1979), and $\alpha \approx 2$ and $\beta \approx 3.85$ for parameters appropriate for mCVs. The dimensionless cooling/heating term $\tilde{\Lambda}$ is then

$$\tilde{\Lambda} \; = \; \left(\frac{x_{\rm s}}{\rho_{\rm a} v_{\rm ff}^3} \right) \left\{ \rho_{\rm a}^2 v_{\rm ff} A \left(\frac{\sigma}{\tau} \right)^{3/2} \sqrt{1 - \tau \sigma} \left[1 \; + \; \epsilon_{\rm s} \frac{4^{\alpha+\beta}}{3^{\alpha}} \left(1 - \tau \sigma \right)^{\alpha} \left(\frac{\tau}{\sigma} \right)^{\beta} \right] \; - \; \Lambda_{\rm h}(\tau) \right\} \; . \tag{13}$$

With the cooling function is defined, the flow velocity, and hence the other hydrodynamic variables, can be derived after equation (11) is inverted.

If we set $\gamma = 5/3$, $\sigma(\tau) = 1$ and $\Lambda_{\rm h} = 0$, equation (11) becomes

$$\xi(\tau) \; = \; \frac{v_{\rm ff}^2}{2 A x_{\rm s} \rho_{\rm a}} \int_0^{\tau} d\tau' \, \frac{\tau'^2 \, (5 - 8\tau')}{\sqrt{\tau'(1 - \tau')}} \left[1 \; + \; \epsilon_{\rm s} \frac{4^{\alpha+\beta}}{3^{\alpha}} \, (1 - \tau')^{\alpha} \tau'^{\beta} \right]^{-1} \; , \tag{14}$$

which is identical to equation (3) in Wu (1994). This limiting case corresponds to "no leaking" from the accretion column and a hard, stationary white-dwarf surface with no emergent energy flux from below. If we further set $\epsilon_{\rm s} = 0$, the Aizu (1973) solution is recovered.

2.2 The sink term

When the accreting material is cooled sufficiently, recombination occurs and neutral atoms are formed. While the charged ionised matter is magnetically confined, neutral matter can diffuse across the field lines. As the temperature is lower near the base of the accretion column, we expect matter leakage to be efficient.

In the hydrodynamic formulation that we consider, the temperature and the flow velocity are not independent. The temperature is a monotonic function of the velocity, and in the limit of no leakage it is linearly proportional to the velocity at the base of the accretion column. The sink (which depends on the hydrodynamic variables) can therefore be chosen to be an explicit function of the flow velocity only. We assume that it is characterised by a critical velocity $\tau_{\rm m}$, below which matter leakage becomes efficient. This is equivalent to assuming a critical temperature at which recombination becomes sufficiently efficient to allow significant amount of neutral matter to form and to diffuse out of the accretion column.

It is more convenient to consider $\sigma(\tau)$ (the specific mass-accretion rate per unit area remaining in the post-shock flow at height ξ after leakage) than the leakage at any height $\tilde{\Sigma}(\xi)$ itself. Suppose $\sigma(\tau)$ can be expanded into a series of orthogonal bases $e^{-n\tau/\tau_{\rm m}}$, such that $\sigma(\tau) = \sum a_n e^{-n\tau/\tau_{\rm m}}$, where a_n is the coefficient of the n-th power term. If only the first two leading terms are important, as an approximation we may consider

$$\sigma(\tau) \; = \; \mu \left(1 - \nu e^{-\tau/\tau_{\rm m}} \right), \tag{15}$$

where μ and ν are constants to be determined. In this study we treat $\tau_{\rm m}$ as a parameter. When appropriate atomic and magneto-hydrodynamic processes are considered, an explicit expression of it in terms of the other system parameters can be obtained.

The functional form of $\sigma(\tau)$ above must satisfy the hydrostatic-equilibrium condition at the white-dwarf surface ($\xi = 0$). This requires the mass flux $\sigma(\tau)$ equal to zero at the white-dwarf surface. Thus, the constant $\nu = 1$. At the shock ($\xi = 1$), the mass flux $\sigma(\tau) = 1$ by definition (equation 7), implying a normalisation constant $\mu = (1 - e^{-1/4\tau_{\rm m}})^{-1}$. In the limit of $\tau_{\rm m} \ll 1$, the velocity-derivative of $\sigma(\tau)$ is

$$\frac{\partial}{\partial \tau} \sigma \approx \frac{1}{\tau_{\rm m}} \tag{16}$$
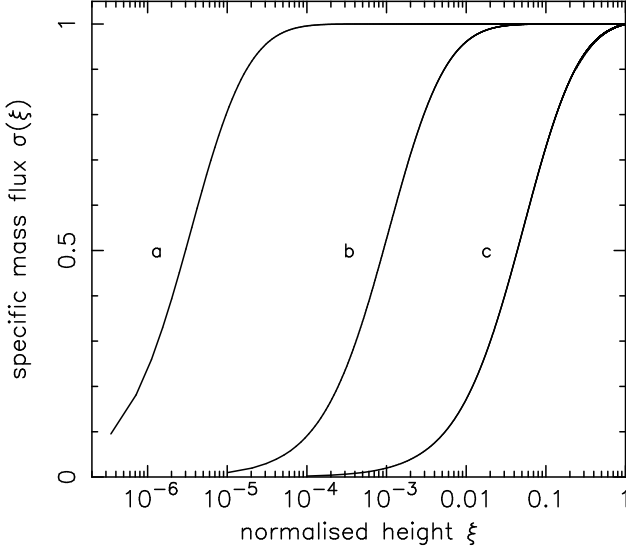
Figure 1. Specific mass flux $\sigma(\xi)$ (mass transfer rate per unit area) as a function of height ξ within the post-shock region for $\tau_m =$ 0.001, 0.01 and 0.05 (curves a, b and c respectively).

at the white-dwarf surface, and it is

$$\frac{\partial}{\partial \tau} \sigma \approx 0 \tag{17}$$

at the shock. Matter leakage therefore occurs only at the very bottom of the accretion column if τ_m is sufficiently small, and mass is practically conserved along the flow in most of the post-shock region. For $\tau_m = 0.01$, the specific mass flux falls to less than 1/2 of its initial value only when the normalised height $\xi \lesssim 0.001$; and for $\tau_m = 0.001$, when $\xi \lesssim 3 \times 10^{-6}$ (Fig. 1).

2.3   Energy flux from the white-dwarf atmosphere

We assume that the heating due to the energy flux from below the white-dwarf surface is important only in a thin region (with a thickness $\Delta \tau < \tau_m$) at the bottom of the accretion column. Suppose $\pi S_w(0)$ is the radiation flux emerging normally from the white-dwarf surface. The radiation is absorbed by the accreting matter and is attenuated. At the height ξ, the radiation flux is given by

$$\pi S_w(\xi) = \pi S_w(0) \exp\left(-x_s \int_0^\xi d\xi' \, \kappa(\xi')\right) \tag{18}$$

where $\kappa(\xi)$ is the absorption coefficient. The rate of energy deposited per unit volume is therefore

$$-\frac{\pi}{x_s}\frac{d}{d\xi} S_w(\xi) = \pi \, \kappa(\xi) S_w(0) \exp\left(-x_s \int_0^\xi d\xi' \, \kappa(\xi')\right) . \tag{19}$$

The absorption coefficient κ depends only on the local density and temperature, which are approximately constant when $\tau < \tau_m$ (see next section). In local thermal equilibrium, the absorption coefficient is determined by the emissivity, which is the effective cooling function in our formulation, and the Planck function $\mathcal{B}$ (Kirchoff's Law). As the emergent flux from the white dwarf is a black-body flux, i.e. $S_w(0) = \mathcal{B}(0)$, we have

$$\kappa(\xi)S_w(0) \approx \kappa(0)S_w(0)$$
$$= \frac{1}{\pi}\Lambda_c(0) . \tag{20}$$

Moreover,

$$\int_0^\xi d\xi' \, \kappa(\xi') = \int_0^\xi d\xi' \left[\frac{\Lambda_c(\xi')}{\mathcal{B}(\xi')}\right]$$
$$\approx \frac{\xi \, \Lambda_c(0)}{\mathcal{B}(0)} . \tag{21}$$

We have defined the lower boundary of the accretion column as the location where the heating rate equals the cooling rate and the flow velocity is zero. This requires $\Lambda_h(\tau) = \Lambda_c(\tau)$ and $\tau = 0$ at $\xi = 0$. Suppose the attenuation factor can be expressed in terms of the dimensionless velocity τ, such that

$$x_s \int_0^{\xi(\tau)} d\xi' \, \kappa(\xi') = \frac{\sqrt{\tau}}{\lambda} + \mathcal{O}(\tau) , \tag{22}$$

where λ is a constant to be determined. Then, the heating function in the boundary layer is

$$\Lambda_h(\tau) \approx \Lambda_c(0) \, \exp\left(-\frac{\sqrt{\tau}}{\lambda} \right) . \tag{23}$$

By expanding τ into Taylor series at the boundary $\xi_o = 0$, we obtain

$$\sqrt{\tau} = \left[\xi \frac{\partial \tau}{\partial \xi}\bigg|_{\xi_o=0} + \frac{\xi^2}{2!} \frac{\partial^2 \tau}{\partial \xi^2}\bigg|_{\xi_o=0} + \ldots\ldots \right]^{1/2} . \tag{24}$$

Smooth merging of the flow into a hydrostatic white-dwarf atmosphere requires the first derivative of the velocity to be zero. By combining equations (10), (23) and (24) and keeping only the first non-vanishing term, we obtain

$$\sqrt{\tau} = \frac{\xi}{2\lambda} \left(\frac{\gamma - 1}{\gamma} \right) \left(\frac{x_s \Lambda_c(0)}{\rho_a v_{\rm ff}^3} \right) . \tag{25}$$

We can now define a characteristic critical velocity τ_h, below which heating by the flux from below the white-dwarf atmosphere is important and beyond which heating effects are negligible. τ_h is given by

$$\begin{aligned}
\tau_h &\equiv \lambda^2 \\
&= \frac{1}{2} \left(\frac{\gamma - 1}{\gamma} \right) \left(\frac{\mathcal{B}(0)}{\rho_a v_{\rm ff}^3} \right) .
\end{aligned} \tag{26}$$

The heating function

$$\Lambda_h(\tau) = \Lambda_c(0) \, \exp\left(-\sqrt{\frac{\tau}{\tau_h}} \right) \tag{27}$$

is asymptotically zero for $\tau \gg \tau_h$. As heating is unimportant for $\tau > \tau_m > \tau_h$, we can simply use the heating function above throughout the whole post-shock region, despite the fact it is derived from considering heating of the accreting material within the boundary layer.

3   PROPERTIES OF THE HYDRODYNAMIC VARIABLES AT THE LOWER BOUNDARY

3.1   Hydrodynamic variables

The matter density at the base of the "leaky" accretion column that we consider is

$$\begin{aligned}
\zeta(0) &= \lim_{\tau \to 0} \frac{\mu}{\tau} \left(1 - e^{-\tau/\tau_m} \right) \\
&= \frac{\mu}{\tau_m} ,
\end{aligned} \tag{28}$$

which is clearly finite. The pressure is

$$\begin{aligned}
\varpi(0) &= \lim_{\tau \to 0} \left[1 - \tau\mu \left(1 - e^{-\tau/\tau_m} \right) \right] \\
&= 1 ,
\end{aligned} \tag{29}$$

the same as that in the conventional formulation. The dimensionless temperature is defined as $\theta \equiv T/T_s$, and hence we have

$$\theta = \frac{\zeta_s}{\varpi_s} \frac{\varpi}{\zeta} . \tag{30}$$

At the white-dwarf surface, the temperature is

$$\begin{aligned}
\theta(0) &= \lim_{\tau \to 0} \frac{16}{3} \frac{\tau}{\mu} \left[\frac{1 - \tau\mu(1 - e^{-\tau/\tau_m})}{1 - e^{-\tau/\tau_m}} \right] \\
&= \frac{16}{3} \frac{\tau_m}{\mu} ,
\end{aligned} \tag{31}$$

which is also finite.

In the limit of $\tau_m \to 0$, we have $\sigma(\tau) \to 1$ and $\partial\sigma/\partial\tau \to 0$. Moreover, $\zeta(0) \to \infty$, $\theta(0) \to 0$, and $\varpi(0) = 1$, i.e. the cold "stationary-wall" boundary condition is recovered.

3.2   Gradients of the hydrodynamic variables

Not only are the hydrodynamic variables of the conventional and our formulations are different at the lower boundary, but they also have very different gradients. In the conventional formulation, the density gradient is given by

$$\frac{\partial}{\partial \xi}\zeta = -\frac{1}{\tau^2}\frac{\partial}{\partial \xi}\tau \; . \tag{32}$$

As $\partial\tau/\partial\xi \propto \tau^{-3/2}$ for small τ (from equation (2) in Wu 1994), the density gradient is proportional to $\tau^{-7/2}$. At $\xi = 0$, $\tau = 0$; hence the density gradient is infinite. The pressure gradient, which is

$$\begin{aligned}\frac{\partial}{\partial \xi}\varpi &= -\frac{\partial}{\partial \xi}\tau \\ &\propto -\tau^{-3/2} \; ,\end{aligned} \tag{33}$$

and the temperature gradient, which is

$$\begin{aligned}\frac{\partial}{\partial \xi}\theta &= \frac{16}{3}(1 - 2\tau)\frac{\partial}{\partial \xi}\tau \\ &\propto -\tau^{-3/2} \; ,\end{aligned} \tag{34}$$

are also infinite at $\xi = 0$;

For a "leaky" accretion column allowing an energy flux emerging from below, the velocity gradient at the base is

$$\begin{aligned}\lim_{\tau \to 0}\frac{\partial}{\partial \xi}\tau &= \lim_{\tau \to 0}(\gamma - 1)\,\tilde{\Lambda}\left[\gamma - (\gamma + 1)\,\tau\sigma - \frac{1}{2}(\gamma + 1)\,\tau^2\frac{\partial}{\partial \tau}\sigma\right]^{-1} \\ &= \left[\frac{(\gamma - 1)x_s}{\gamma \rho_a v_{ff}^3}\right](\Lambda_c(0) - \Lambda_h(0)) \; .\end{aligned} \tag{35}$$

The velocity gradient is zero at $\xi = 0$, provided that $\Lambda_c(0) = \Lambda_h(0)$. This is in fact the condition of radiative equilibrium in a hydrostatic white-dwarf atmosphere.

Differentiating equation (8) with respect to ξ yields the density gradient:

$$\frac{\partial}{\partial \xi}\zeta = -\frac{1}{\tau}\left[\frac{\sigma}{\tau} - \frac{\partial}{\partial \tau}\sigma\right]\frac{\partial}{\partial \xi}\tau \; . \tag{36}$$

By substituting the expression of $\sigma(\tau)$ given in equation (15) into the equation above, we obtain

$$\frac{\partial}{\partial \xi}\zeta = \frac{\mu}{\tau_m^2}\left[\sum_{n=0}^{\infty}C_n\left(\frac{\tau}{\tau_m}\right)^n\right]\frac{\partial}{\partial \xi}\tau \quad (\tau < \tau_m) \; , \tag{37}$$

where

$$C_n = (-1)^n\left[\frac{1}{(n + 1)!} - \frac{1}{(n + 2)!}\right] \; . \tag{38}$$

The power series C_n converges, with a value of the order of unity when $\tau \to 0$ (Appendix A.1). The density gradient is therefore finite and equals zero at $\xi = 0$.

The pressure gradient is

$$\frac{\partial}{\partial \xi}\varpi = -\mu\left[1 - \left(1 - \frac{\tau}{\tau_m}\right)e^{-\tau/\tau_m}\right]\frac{\partial}{\partial \xi}\tau \; . \tag{39}$$

It is also finite and equals zero at $\xi = 0$. The temperature gradient is

$$\frac{\partial}{\partial \xi}\theta = \frac{\zeta_s}{\varpi_s}\left[\frac{1}{\zeta^2}\left(\zeta\frac{\partial}{\partial \tau}\varpi - \varpi\frac{\partial}{\partial \tau}\zeta\right)\right]\frac{\partial}{\partial \xi}\tau \; . \tag{40}$$

As $\partial\varpi/\partial\tau$ and $\partial\zeta/\partial\tau$ are both finite and the density ζ is non-zero at $\xi = 0$, the temperature gradient is zero.

4   ACCRETION LUMINOSITY

In the conventional formulation, energy conservation holds along the field (flow) line, so that the accretion energy must be radiated away before the accreting matter can settle down onto the white-dwarf surface. The accretion luminosity (normalised to $\rho_a v_{ff}^2$) along the flow line down to the height ξ is

$$L_{acc}(\xi, 1) = \int_{\xi}^{1}d\xi\,\tilde{\Lambda}$$

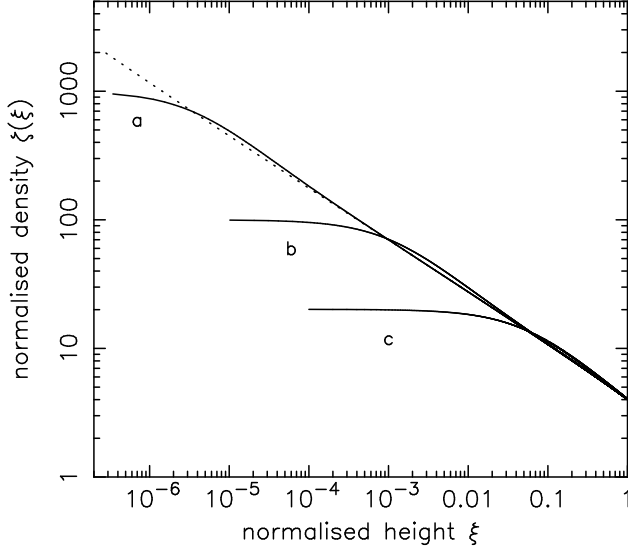**Figure 2.** Density $\zeta(\xi)$ as a function of height ξ for $\tau_{\mathrm{m}} = 0.001$, 0.01 and 0.05 (curves a, b and c respectively). The density dependence of the conventional model (Aizu 1973) is also shown for comparison (dotted line).

$$
\begin{aligned}
&= \frac{1}{\gamma - 1} \int_{\tau}^{1/4} d\tau \; [\gamma - (\gamma + 1)\tau] \\
&= \frac{1}{\gamma - 1} \left\{ \left[\frac{\gamma}{4} - \frac{1}{32}(\gamma + 1) \right] - \gamma\tau \left[1 - \left(\frac{\gamma + 1}{2\gamma} \right)\tau \right] \right\} .
\end{aligned}
\tag{41}
$$

The total bolometric accretion luminosity is therefore

$$
L_{\mathrm{acc}}(0, 1) = \frac{1}{\gamma - 1} \left[\frac{\gamma}{4} - \frac{1}{32}(\gamma + 1) \right] .
\tag{42}
$$

For $\gamma = 5/3$, $L_{\mathrm{acc}}(0, 1) = 1/2$, consistent with the fact that the total accretion energy liberated is independent of the nature of the cooling process.

For the "leaky" accretion column considered here, energy conservation also holds strictly along the flow lines. The bolometric accretion luminosity along a field line down to ξ is

$$
\begin{aligned}
L_{\mathrm{acc}}(\xi, 1) &= \frac{1}{\gamma - 1} \int_{\tau}^{1/4} d\tau \left[\gamma - (\gamma + 1)\tau\sigma - \frac{1}{2}(\gamma + 1)\tau^2 \frac{d\sigma}{d\tau} \right] \\
&= \frac{1}{\gamma - 1} \left\{ \left[\frac{\gamma}{4} - \frac{1}{32}(\gamma + 1) \right] - \gamma\tau \left[1 - \left(\frac{\gamma + 1}{2\gamma} \right) \left(\frac{1 - e^{-\tau/\tau_{\mathrm{m}}}}{1 - e^{-1/4\tau_{\mathrm{m}}}} \right) \tau \right] \right\}
\end{aligned}
\tag{43}
$$

(Appendix A.2), where σ is as defined in equation (15), and the total bolometric accretion luminosity is

$$
L_{\mathrm{acc}}(0, 1) = \frac{1}{\gamma - 1} \left[\frac{\gamma}{4} - \frac{1}{32}(\gamma + 1) \right] .
\tag{44}
$$

For $\gamma = 5/3$ it is equal to 1/2, consistent with the requirement of strict energy conservation along a field line.

The bolometric accretion luminosity of the boundary layer, where $\tau < \tau_{\mathrm{m}}$, is

$$
\begin{aligned}
L_{\mathrm{acc}}(0, \xi_{\mathrm{m}}) &= L_{\mathrm{acc}}(0, 1) - L_{\mathrm{acc}}(\xi_{\mathrm{m}}, 1) \\
&= \left(\frac{\gamma\,\tau_{\mathrm{m}}}{\gamma - 1} \right) \left[1 - \frac{\gamma + 1}{2\gamma} \left(\frac{1 - e^{-1}}{1 - e^{-1/4\tau_{\mathrm{m}}}} \right) \tau_{\mathrm{m}} \right] ,
\end{aligned}
\tag{45}
$$

where ξ_{m} is the height at which $\tau = \tau_{\mathrm{m}}$. If the boundary layer is thin (i.e. $\tau_{\mathrm{m}} \ll 1$), the bolometric luminosity $\sim \gamma\tau_{\mathrm{m}}$, which is insignificant in comparison to that of the rest of the post-shock region.
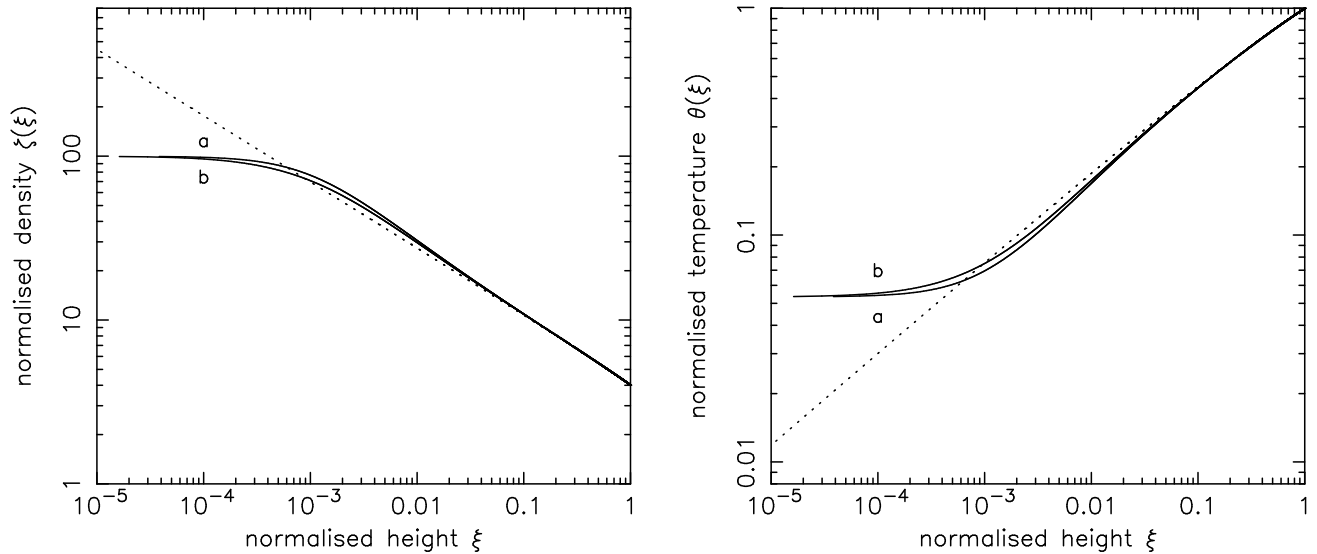
5 DISCUSSION

Figure 3. Density (left) and temperature (right) as a function of height ξ for heating scale height $\tau_\mathrm{h} = 0.001$ and 0.0001 (curves a and b respectively). The leakage parameter τ_m is fixed to be 0.01. The conventional model is shown (in dotted line) for comparision.

5.1   Structure of the post-shock region

Figure 2 shows the density profile at the base of the post-shock region for "leaky" accretion columns with different values of τ_m by comparison with the conventional formulation (Aizu 1973), in which the lower boundary is a cold, stationary wall. As noted above, the density ζ and its gradient in the conventional formulation approach infinity at the base of the accretion column, whereas the density in our formulation has a finite value, μ/τ_m. For $\tau_\mathrm{m} = 0.05$, 0.01 and 0.001, $\zeta(0) \approx 20$, 100 and 1000 respectively. In all cases of τ_m, the density at the "leaky" base falls significantly below that of the base in the conventional formulation for ξ less than ξ_m, the height at which $\tau = \tau_\mathrm{m}$.

In calculating the density profiles in Figure 2, we have not included the effect due to heating from the white dwarf, i.e. we have considered $\tau_\mathrm{h} = 0$. We now show this effect in Figure 3, where we have fixed the critical velocity τ_m to 0.01, and calculated the dimensionless density and temperature for two different boundary layers, with $\tau_\mathrm{h} = 0.001$ and 0.0001. They both fulfill the constraint that $\tau_\mathrm{h} < \tau_\mathrm{m}$. As shown, the terminated temperature depends mainly on τ_m. The heat input from the white dwarf modifies only the asymptotic properties of the base temperature — a larger τ_h will result in a slightly thicker "isothermal" layer at the base, thus reducing the temperature upstream and causing the density to reach the terminated plateau value further upstream.

Figure 4 shows the dimensionless local bremsstrahlung emissivity j_br ($= \zeta^2 \theta^{1/2}$) as a function of the height ξ in the post-shock region. The emissivity reaches a terminated value determined by τ_m, whereas that from the conventional formulation tends to infinity. Again the bremsstrahlung emissivity is modified slightly by the input of heat from the white dwarf when $\tau_\mathrm{h} > 0$ with increased emission for larger values of τ_h. This is in fact the consequence of an implicit assumption that the white-dwarf energy flux is much weaker than energy flux released by accretion material.

5.2   Optical depths

Using the prescription for density and temperature above we can easily show that the optical depth due to free-free absorption or electron scattering in the vertical upstream direction is negligible when the photon energies are above 0.05 keV, for typical mass-transfer rates and white-dwarf masses. However, by assigning the parameter τ_h for the heating at the base of the region we implicitly assume a large opacity to maintain a black-body spectrum for the radiative flux. As the temperature in the region of our interest should be sufficiently low such that recombination can occur to form neutral atoms, there are additional sources of opacity, such as the bound-free and bound-bound opacities, to maintain a large optical depth in this geometrically thin layer at the base.

The bound-free and bound-bound emission processes are, however, not considered explicitly in constructing the heating and cooling functions in our formulation. Instead, we consider only a parametric heating function Λ_h, which is in terms of the accretion parameters and the energy flux from below the white-dwarf surface. The formulation that we present in this work is therefore not a fully self-consistent formulation with explicitly consideration of the microscopic atomic physics (see van Teeseling, Heise & Paerels 1994).

Also, we have not considered an explicit treatment of the ionisation equilibrium. There might be situations that the gas at
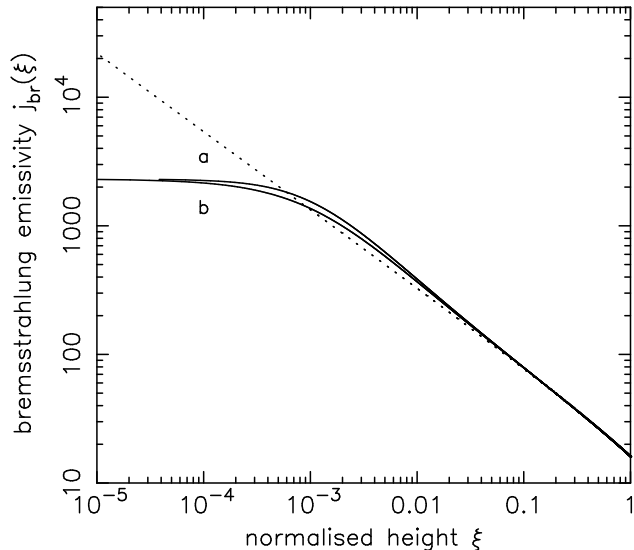
Figure 4. Bolometric bremsstrahlung emissivity j_{br} ($= \zeta^2 \theta^{1/2}$) as a function of height ξ. Curve a corresponds to the case of $\tau_m = 0.01$ and $\tau_h = 0.001$; curve b to the case of $\tau_m = 0.01$ and $\tau_h = 0$. The bremsstrahlung emissivity of the conventional model is shown (in dotted line) for comparison.

the base is photo-ionised by the X-ray from the shock above. The increase in the degree of ionisation will cause an decrease in the efficiency of matter diffusion across the field lines, and hence out of the accretion column to the white-dwarf atmosphere. For typical mCV parameters, say white-dwarf mass $M_{wd} = 0.7$ M$_\odot$, specific mass-accretion rate $\dot{m} \sim 1$ g cm^{-2}s^{-1}, X-ray luminosity L_x of $\sim 10^{32}$ erg s^{-1}, shock temperature $T_s \sim 10$ keV and shock height $x_s \sim 3 \times 10^7$ cm, the ionisation parameter Ξ ($\equiv L_x/nr^2$, where n is the number density and r is the distance to the X-ray source) has values $\lesssim 0.01$. (Here, we have assumed $\tau_m \approx 5.0 \times 10^{-5}$ in the calculation of Ξ.) The value of Ξ is significantly less than 20, the critical value at which hydrogen becomes completely ioinised (see Models 3 and 4 in Kallman & McCray 1982). Hence, the formulation presented above is in general applicable to the accretion column of mCVs. Such a parametric prescription allows to derive a analytic formulation to describe the accretion column and the boundary layer in a simple and unified manner.

It is worth noting that the surface at which the total scattering and absorption optical depth is unity is in effect the observable lower boundary of the post-shock region. This surface does not necessarily coincide with the base as that defined by the flow hydrodynamics, the zero-velocity surface. The emission from this unit-optical-depth surface will be observed as soft X-rays with a blackbody-like spectrum characterised by a temperature similar to that of the isothermal layer (due to $\tau_h < \tau_m$).

5.3   X-ray spectral properties

With the density and temperature specified within the post-shock region we can calculate the emission at different heights and construct integrated spectra. We consider the case in which bremsstrahlung radiation is the only cooling process and ignore cyclotron cooling, so to illustrate the differences between between it and the conventional post-shock X-ray emission regions resulting from the Aizu (1973) formulation.

Figure 5 illustrates the differences in the emitted spectrum with respect to the conventional formulation for 1.0-M$_\odot$ and 0.5-M$_\odot$ white dwarfs. The variable $\psi_{bf}(E)$ is the ratio of the bremsstrahlung emissivity for the "leaky" post-shock region $j_{br}(E; \tau_m)$ to that for the conventional region $j_{br}(E; 0)$, where E is the photon energy. When τ_m is relatively large (~ 0.01), a significant fraction of accreting material is lost from the post-shock region at large height (see Fig. 1). Because we require all of the accretion energy to be liberated before the matter diffuses out of the post-shock region, the accretion energy is radiated at higher energies and the 1–20 keV region of the resulting spectrum is enhanced with respect to that of the spectrum from the conventional post-shock region.

For 1.0-M$_\odot$ white dwarfs the spectrum below ~ 5 keV is harder than that from the conventional post-shock region, while above this value it is softer. This effect is enhanced if significant heat flux is permitted from the white dwarf (larger values of τ_h). For lower values of τ_m, the temperatures are already low at the base of the region, and the main effect is that of the increased heating from the white dwarf at energies below 1 keV. The spectrum in this case is softer than that from the conventional formulation, but only at low energies. When both τ_m and τ_h tend to zero, the spectrumretains the conventional form.
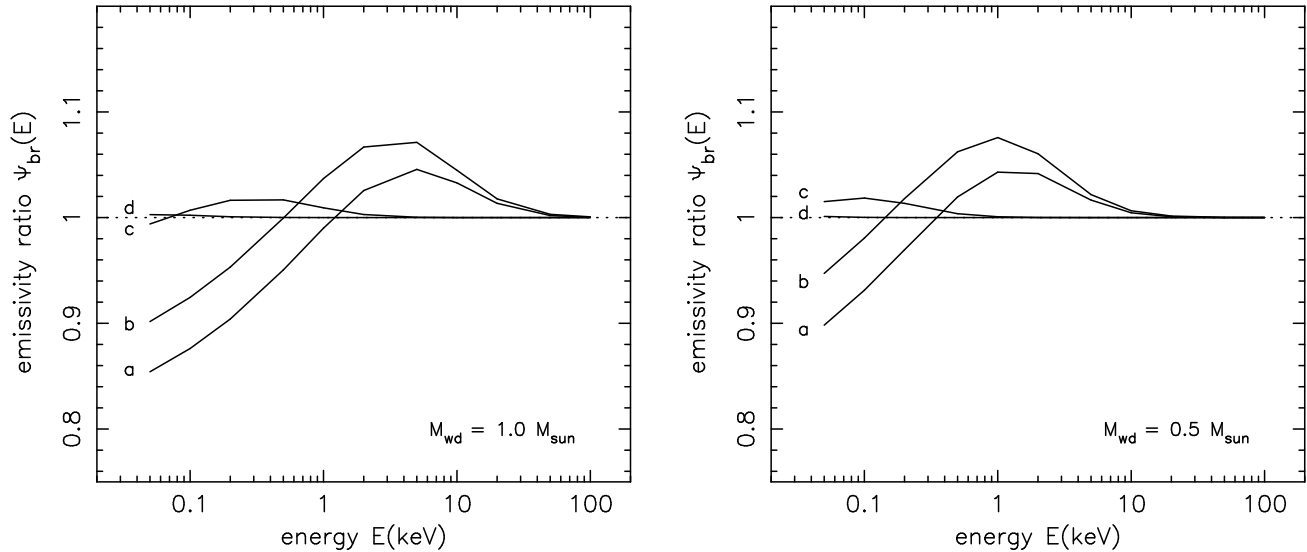
Figure 5. The ratio of total bremsstrahlung emissivity of the "leaky" accretion column to that of the conventional accretion column ψ_{br} as a function of photon energy E for a 1.0-$M_{\odot}$ white dwarf (left) and a 0.5-$M_{\odot}$ white dwarf (right). The leakage and heating parameters are: $(\tau_{\mathrm{m}}, \tau_{\mathrm{h}}) = (0.01,0)$, $(0.01,0.001)$, $(0.001,0.0001)$ and $(0.0001,0.00001)$ for cuves a, b, c and d respectively. The specific mass flux is 1 g s^{-1}cm^{-2} at the shock ($\xi = 1$).

5.4   Observational consequences

From Figure 5 it is clear that it is possible to obtain either a harder or softer spectrum than that from the conventional post-shock region, depending on leakage parameter τ_{m} and heating parameter τ_{h}, and on the energy range considered. When fitting to X-ray spectral data, a harder model spectrum will result in a lower-mass determination for the white dwarf. If τ_{m} is significant, then for typical CCD or proportional counter detectors operating in the 0.2–20 keV range, fitting spectra from the "leaky" formulation will generally therefore result in lower masses for more massive white dwarfs than those obtained using the Aizu (1973) or Wu et al. (1994) formulations. In the case of lower-mass white dwarfs the effect is to mimic the change in spectral slope resulting from increased cyclotron cooling from a strong magnetic field.

It should be noted that because they are parameterisations, the treatments in §2.2 and §2.3 do not provide a prescription for definite values of τ_{m} and τ_{h} based on atomic physics and magneto-hydrodynamics. However, if we simply assume certain values for the effective temperature T_{eff} of the white-dwarf atmosphere, then we can obtain an estimate for τ_{h} by equation (26) after the specific accretion rates (or the specific pre-shock mass flux) $\dot{m}$ and the white-dwarf mass M_{w} are specified. If we further assume that the temperature $T(0)$ at the boundary (i.e. where τ and its derivative is zero) as the unperturbed effective temperature of the white-dwarf atmosphere, then we may also obtain an estimate for τ_{m} using equation (31). In reality, we expect $T(0)$ to be larger because of radiative heating by the X-rays from the post-shock region above and T_{s} to be less than that determined by the free-fall velocity at the shock surface (see Cropper et al. 1999). Therefore, we have underestimated τ_{m} here, and the value that we obtained can be considered as a lower limit to τ_{m}. Nevertheless, under this simple approximation, we have found that the assumption $\tau_{\mathrm{h}} < \tau_{\mathrm{m}}$ is acceptable, provided that T_{eff} is not very significantly larger than 10^4 K. Moreover, the assumption that $\tau_{\mathrm{m}} \ll 1$ is in general valid. This is shown in Figure 6.

Which values of τ_{m} can be reached is therefore unclear pending further investigation, but the values indicated from Figure 6 are in the range 10^{-4} to 10^{-5}, significantly lower than those used in Figure 5. The difference between spectra from models with these smaller values of τ_{m} and those from the Aizu (1973) formulation is less than 1% in the range of 0.05 eV to 100 keV, so that the observational effects of the leaky base will benegligible. Thus if we can indeed rely on small values of τ_{m}, current cold wall models such as those mentioned in Section 1 can continue to be employed for X-ray spectral fitting.

The advantage of our treatment is that it avoids the infinities at the base of the flow inherent in all previous formulations in which the stationary-wall boundary condition base is assumed (e.g. Aizu 1973; Chevalier & Imamura; Wu 1994; Wu et al. 1994). This is a step in the direction of more realistic formulations which allow the hydrodynamic accretion flow to match the hydrostatic white-dwarf atmosphere. As noted above the predicted spectral changes are small for small τ_{m}; however a *practical* advantage is that it eliminates the numerical errors resulting from finite sampling of quantities which tend to infinity. In the previous treatments the fineness of the sampling of the base of the post-shock region in the numerical integrating scheme has an effect on the spectral slope: finer sampling will encounter a higher value of the density (which is tending to infinity) at the base, so that softer spectra will be generated. In our formulation, as the singularity at the base is removed, the effect of the sampling is insignificant even when small values of τ_{m} are assumed. This is consistent with the fact that emission from the boundary layer should be finite, as deduced from equation (45).
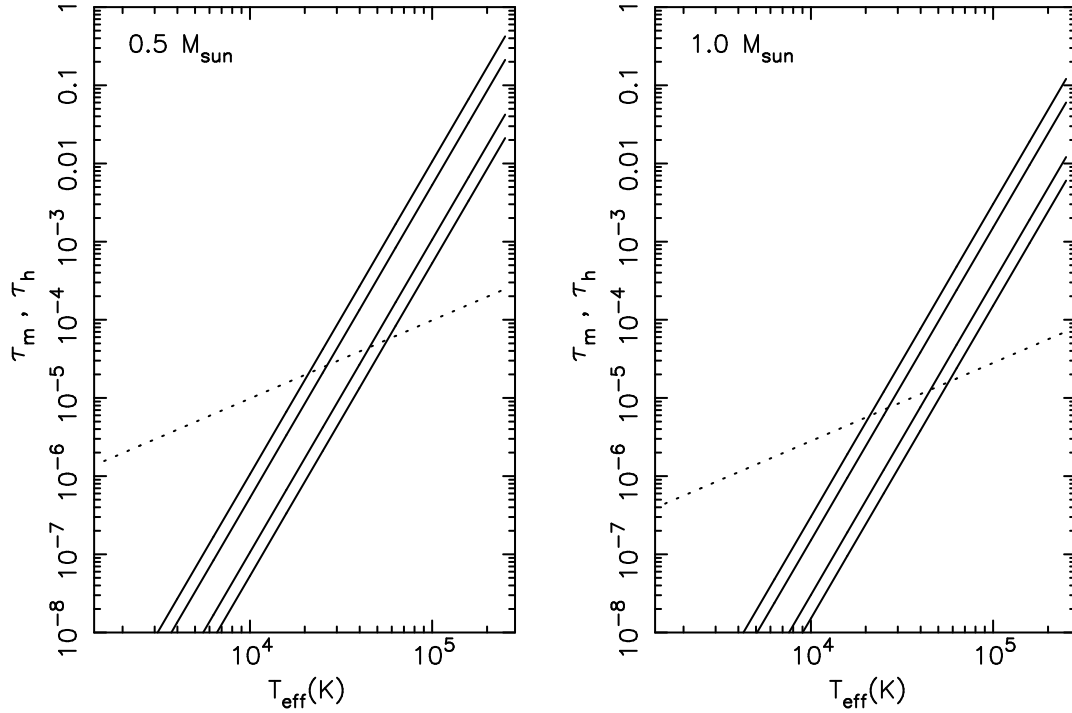
Figure 6. The values of τ_m (dotted lines) and τ_h (solid lines) as a function of the effective temperature of the white-dwarf atmosphere T_{eff} for 0.5- and 1.0-$M_\odot$ white dwarfs (left and right panel respectively). The values of specfic accretion rates to calculate τ_h are 0.5, 1.0, 5.0 and and 50.0 g cm^{-2}s^{-1} (solid lines from left to right). Here, we have considered the mass-radius relation of white dwarfs given in Nauenberg (1972) and assumed that the temperature at the boundary $T(0)$ is the same as T_{eff}. The shock temperature T_s is calculated by assuming a strong adiabatic-shock condition and a cosmic abundance.

6   ACKNOWLEDGEMENTS

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A.1 Convergence of the power series $\sum C_n x^n$

Consider a power series

$$f(x) \;=\; \sum_{n=0}^{\infty} C_n x^n \qquad (x < 1),$$

where

$$C_n \;=\; (-1)^n \left[\frac{1}{(n+1)!} - \frac{1}{(n+2)!} \right].$$

The series is convergent for $x < 1$ if the ratio of the coefficients $|C_{n+1}|/|C_n| \le 1$ for sufficiently large n. As

$$\lim_{n\to\infty} \frac{|C_{n+1}|}{|C_n|} \;=\; \lim_{n\to\infty} \frac{(n+2)}{(n+1)(n+3)}$$
$$=\; 0\,,$$

the convergent radius of $f(x)$ is infinite. Moreover, for $x > 0$

$$|f(x)| \;\le\; \sum_{n=0}^{\infty} |C_n| x^n$$
$$=\; \sum_{n=0}^{\infty} \frac{1}{(n+2)} \frac{x^n}{n!}$$
$$<\; e^x\,.$$

Hence, $|f(x)| < 1$ for $x \to 0^+$.

Appendix A.2 Accretion luminosity

For the "leaky" accretion column with a specific mass flux $\sigma = \mu(1 - e^{-\tau/\tau_{\mathrm{m}}})$, where $\mu = (1 - e^{-1/4\tau_{\mathrm{m}}})^{-1}$, the accretion luminosity is

$$L_{\mathrm{acc}}(\xi,1) \;=\; \frac{1}{\gamma-1} \int_{\tau(\xi)}^{1/4} d\tau \left[\gamma - (\gamma+1)\,\tau\sigma - \frac{1}{2}(\gamma+1)\,\tau^2 \frac{d\sigma}{d\tau} \right]$$

$$=\; \frac{1}{\gamma-1} \int_{\tau(\xi)}^{1/4} d\tau \left[\gamma - \mu(\gamma+1)\tau + \mu(\gamma+1)\tau\,e^{-\tau/\tau_{\mathrm{m}}} - \frac{1}{2}(\gamma+1)\left(\frac{\tau^2}{\tau_{\mathrm{m}}}\right) e^{-\tau/\tau_{\mathrm{m}}} \right]$$

$$=\; \frac{1}{\gamma-1} \left[I_1(\xi,1) - I_2(\xi,1) + I_3(\xi,1) - I_4(\xi,1) \right],$$

where

$$I_1(\xi,1) \;=\; \gamma \int_{\tau(\xi)}^{1/4} d\tau$$
$$=\; \gamma \left[\frac{1}{4} - \tau \right],$$

$$I_2(\xi,1) \;=\; \mu(\gamma+1) \int_{\tau(\xi)}^{1/4} d\tau\,\tau$$
$$=\; \frac{\mu}{2}(\gamma+1) \left[\frac{1}{16} - \tau^2 \right],$$

$$I_3(\xi,1) \;=\; \mu(\gamma+1) \int_{\tau(\xi)}^{1/4} d\tau\,\tau\,e^{-\tau/\tau_{\mathrm{m}}}$$
$$=\; \mu(\gamma+1)\tau_{\mathrm{m}}^2 \left[\left(1 + \frac{\tau}{\tau_{\mathrm{m}}}\right) e^{-\tau/\tau_{\mathrm{m}}} - \left(1 + \frac{1}{4\tau_{\mathrm{m}}}\right) e^{-1/4\tau_{\mathrm{m}}} \right],$$

$$I_4(\xi,1) \;=\; \frac{\mu}{2}(\gamma+1) \int_{\tau(\xi)}^{1/4} d\tau\,\frac{\tau^2}{\tau_{\mathrm{m}}}\,e^{-\tau/\tau_{\mathrm{m}}}$$
$$=\; \mu(\gamma+1)\tau_{\mathrm{m}}^2 \left[\left(1 + \frac{\tau}{\tau_{\mathrm{m}}} + \frac{\tau^2}{2\tau_{\mathrm{m}}^2}\right) e^{-\tau/\tau_{\mathrm{m}}} \left(1 + \frac{1}{4\tau_{\mathrm{m}}} + \frac{1}{32\tau_{\mathrm{m}}^2}\right) e^{-1/4\tau_{\mathrm{m}}} \right].$$

Hence, we have

$$
\begin{aligned}
L_{\mathrm{acc}}(\xi,1) &= \frac{1}{\gamma-1}\left\{\frac{\gamma}{4} - \frac{\mu}{32}(\gamma+1) - \gamma\tau + \frac{\mu}{2}(\gamma+1)\tau^2 + \mu(\gamma+1)\tau_{\mathrm{m}}^2\left[\left(1+\frac{\tau}{\tau_{\mathrm{m}}}\right) - \left(1+\frac{\tau}{\tau_{\mathrm{m}}}+\frac{\tau^2}{2\tau_{\mathrm{m}}^2}\right)\right]e^{-\tau/\tau_{\mathrm{m}}}\right.\\
&\qquad\left. -\mu(\gamma+1)\tau_{\mathrm{m}}^2\left[\left(1+\frac{1}{4\tau_{\mathrm{m}}}\right) - \left(1+\frac{1}{4\tau_{\mathrm{m}}}+\frac{1}{32\tau_{\mathrm{m}}^2}\right)\right]e^{-1/4\tau_{\mathrm{m}}}\right\}\\
&= \frac{1}{\gamma-1}\left\{\frac{\gamma}{4} - \frac{\mu}{32}(\gamma+1)\left[1-e^{-1/4\tau_{\mathrm{m}}}\right] - \gamma\tau\left[1-\frac{\mu(\gamma+1)}{2\gamma}\left(1-e^{-\tau/\tau_{\mathrm{m}}}\right)\tau\right]\right\}\\
&= \frac{1}{\gamma-1}\left\{\frac{\gamma}{4} - \frac{1}{32}(\gamma+1) - \gamma\tau\left[1-\frac{\gamma+1}{2\gamma}\left(\frac{1-e^{-\tau/\tau_{\mathrm{m}}}}{1-e^{-1/4\tau_{\mathrm{m}}}}\right)\tau\right]\right\}\,.
\end{aligned}
$$